

# Methods for hyperbolic systems with stiff relaxation

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## SUMMARY

Three methods are analysed for solving a linear hyperbolic system that contains stiff relaxation. We show that the semi-discrete discontinuous Galerkin method, with a linear basis, is accurate when the relaxation time is unresolved (asymptotic preserving—AP). The two other methods are shown to be non-AP. To discriminate between AP and non-AP methods, we argue that in the limit of small relaxation time, one should fix the dimensionless parameters that characterize the near-equilibrium limit. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: hyperbolic systems; stiff relaxation; discontinuous Galerkin; finite-volume methods

## 1. INTRODUCTION

Hyperbolic systems with stiff relaxation terms remain a challenge for numerical methods [1–3]. We are interested in the subset of such systems that have a Chapman–Enskog behaviour, such as models for multiphase flow, rarefied gas dynamics, and radiation hydrodynamics. In an effort to better understand the behaviour of numerical methods for these systems, this study will focus on a simple model problem [2, 4]:

$$\partial_{\tilde{t}}\tilde{u} + \partial_{\tilde{x}}\tilde{v} = 0 \quad (1a)$$

$$\partial_{\tilde{t}}\tilde{v} + c_f^2\partial_{\tilde{x}}\tilde{u} = (c_e\tilde{u} - \tilde{v})/\tau \quad (1b)$$

where  $\tau \geq 0$ ,  $c_f^2 > c_e^2$ , and the notation  $(\tilde{\cdot})$  is used to emphasize that a variable is dimensional. We seek numerical methods for (1) that are accurate for all values of  $\tau$ . Of particular difficulty is whenever  $\tau$  is small, where many methods require unreasonable mesh resolution in order to obtain accurate solutions. A Chapman–Enskog expansion for small  $\tau$  shows that the long-time, asymptotic behaviour of  $\tilde{u}(\tilde{x}, \tilde{t})$  is described by an advection–diffusion equation:

$$\partial_{\tilde{t}}\tilde{u} + c_e\partial_{\tilde{x}}\tilde{u} - \tau(c_f^2 - c_e^2)\partial_{\tilde{x}}^2\tilde{u} = 0 \quad (2)$$

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Contract/grant sponsor: Los Alamos National Laboratory, U.S. Department of Energy; contract/grant number: W-7405-ENG-36. LAUR 0011-1301

The main purpose of this study is to define the manner in which a numerical method for (1) should share this same asymptotic behaviour. The analogy here with gas kinetics is that the system (1) corresponds to a Boltzmann closure,  $\tau c_f$  to the mean-free-path, and Equation (2) to the Navier–Stokes equations.

Once the initial condition and suitable boundary conditions have been specified, the solution of system (1) is characterized by two independent dimensionless parameters. Following Reference [1], in this study we use

$$r = c_e/c_f \quad \text{and} \quad \varepsilon = \tau c_f/L \quad (3)$$

where  $L$  is the length scale of interest. On the other hand, solutions to (2) are characterized by a single parameter, the Peclet number, given by

$$P_e = \frac{c_e L}{\tau(c_f^2 - c_e^2)} = \frac{r}{\varepsilon(1 - r^2)} \quad (4)$$

We contend that if conditions are such that (2) holds, then a numerical method for (1) should only have to resolve the length and time scales corresponding to (2). In other words, if  $\varepsilon \ll 1$  and the solution is near equilibrium, then the accuracy of a method should only depend on the resolution of scales related to  $P_e$ , and not necessarily require resolution of scales corresponding to either  $r$  or  $\varepsilon$ . We refer to such methods as *asymptotic preserving* (AP), a term borrowed from [5].

In terms of a characteristic mesh spacing  $\Delta\tilde{x}$  and with  $\varepsilon \ll 1$ , let the degree of mesh resolution be measured as

$$h \equiv \Delta\tilde{x}/L = O(\varepsilon^p) \quad (5)$$

Assume that  $\Delta\tilde{x}$  is no larger than the length scale of interest, so that the minimum  $p$  is 0. In this study, we also restrict  $p$  to integer values. At a minimum, there are three regimes that must be considered when analysing a method [6]:

1. *Unresolved regime*:  $p = 0$ . By ‘unresolved’ we refer only to the time and length scales related to  $\tau$ ; note that  $\Delta\tilde{x}/(c_f\tau) = O(\varepsilon^{-1})$ . Referred to as the ‘thick regime’ in Reference [6].
2. *Intermediate regime*:  $p = 1$ , so that  $\Delta\tilde{x}/(c_f\tau) = O(1)$ .
3. *Resolved regime*:  $p \geq 2$ . In this case, the mesh resolves the relaxation scales and  $\Delta\tilde{x}/(c_f\tau) = O(\varepsilon^{p-1})$ . Referred to as the thin regime in Reference [6].

We may now define an AP method more concretely. Consider the following non-dimensionalization:

$$x = \tilde{x}/L, \quad t = \tilde{t}/\tilde{t}_{\text{ref}}, \quad v = \tilde{v}/\tilde{v}_{\text{ref}}, \quad u = \tilde{u}/\tilde{u}_{\text{ref}} \quad (6)$$

where  $\tilde{t}_{\text{ref}}$ ,  $\tilde{v}_{\text{ref}}$ , and  $\tilde{u}_{\text{ref}}$  are constants to be specified later. Expand  $z \in \{u, v\}$  as

$$z(x, t) = z^{(0)}(x, t) + \varepsilon z^{(1)}(x, t) + O(\varepsilon^2) \quad (7)$$

For small  $\varepsilon$ , we choose (6) such that the leading order solution satisfies

$$\partial_t u^{(0)} + \partial_x u^{(0)} - \frac{1}{P_e} \partial_x^2 u^{(0)} = 0 \quad (8)$$

which is a dimensionless version of (2). We define an AP method as a consistent discretization of (1), such that for all  $p \geq 0$  and small  $\varepsilon$ , the discretization for  $u^{(0)}(x, t)$  is consistent with (8).

We stress that our scaling is such that both the advection *and* diffusion terms appear in the leading order solution. Similar asymptotic arguments, along with an approach to solve efficiently the stiffness arising when  $c_f$  and  $c_e$  are disparate, are covered by Naldi and Pareschi [7]. The ability to predict accurate diffusion, and not just advection, is critical if a method for (1) is to be accurate for *all* values of  $\varepsilon$  and  $r$ . On the other hand, we hesitate to refer to an AP method as ‘uniformly accurate’ [8]. In our opinion, uniform accuracy should imply accuracy under every conceivable asymptotic behaviour, a claim we are not ready to make.

The AP concept extends to more complicated systems. For gas kinetics, if the Knudsen number is small such that the Navier–Stokes equations hold, then an AP discretization of the Boltzmann equation will require resolution of only the scales of interest corresponding to the Navier–Stokes equations. Specifically, an AP method allows the mean-free path to be unresolved, unless that is the length scale of interest (such as in a shock transition layer).

Following Reference [4], a useful analysis technique is to study the asymptotics of the modified (or ‘equivalent’) equation for (1). The asymptotic analysis yields what we refer to as the asymptotic modified equation, which for our model problem takes the form

$$\partial_t u + \partial_x u - \frac{1}{P_e} \partial_x^2 u = \text{TE}(u, \varepsilon, r, h) \quad (9)$$

where  $\text{TE}(u, \varepsilon, r, h)$  is defined as the asymptotic truncation error. All of the methods in this study satisfy  $\text{TE} = O(h^2)$ , but may be non-AP as a result of  $O(h^3)$  terms. When deriving (9) for a particular method, we will assume that  $h$  is small enough to resolve the variation in the unknowns and that the solution is regular enough that a Taylor series is valid. However, we stress that  $\tau$  may be unresolved by the mesh.

We will not review the modified equation or asymptotic analyses here, as both techniques are very well known [4, 6, 7]. To keep this report concise, we will also typically omit the steps in deriving (9) for each method. Boundary and initial conditions should also be considered, but we leave this analysis for future work.

This study analyses two semi-discrete methods and one fully discrete method. In the resolved regime, standard error analyses apply, and each method is second-order accurate for smooth data. For the semi-discrete methods, we do not analyse any particular time integrator, and demonstrate that it is the spatial operator that dominates the asymptotic behaviour. We actually prefer fully discrete methods, but have concentrated on semi-discrete methods because it emphasizes that the difficulties arise independent of the time integration scheme [4, 7]. On the other hand, the spatial operator from a non-AP semi-discrete method, when used with a clever choice of predictor step(s), may result in an AP method [8]. We leave the investigation of such methods for future work.

## 2. ACCURACY OF METHODS FOR $c_e = 0$

Methods may be eliminated from consideration by first studying the  $c_e = 0$  case. This special case simplifies the analysis considerably and elucidates why certain methods fail. In the scaling

(6), we set

$$\tilde{t}_{\text{ref}} = L/(\varepsilon c_f), \quad \tilde{v}_{\text{ref}} = \varepsilon c_f \tilde{u}_{\text{ref}}, \quad \tilde{u}_{\text{ref}} = O(1) \quad (10)$$

System (1) then reduces to

$$\partial_t u + \partial_x v = 0 \quad (11a)$$

$$\partial_t v + \frac{1}{\varepsilon^2} \partial_x u = -\frac{1}{\varepsilon^2} v \quad (11b)$$

If we substitute (7) and collect the  $O(\varepsilon^0)$  terms, then we find

$$\partial_t u^{(0)} - \partial_x^2 u^{(0)} = 0 \quad (12)$$

The motivation for scaling (10) is now apparent, so that the leading-order asymptotic solution is the heat equation. Naldi and Pareschi [7] refer to this particular scaling as *diffusive*. To leading order, the discretization of an AP method for (11) must be consistent with (12).

The Fourier transform solution of (12) shows that data of wave number  $k$  is damped as  $\exp(-dt)$ , where  $d$  is the damping rate, given by  $d = 4\pi^2 k^2$ . Let  $d_h$  be the damping rate for a particular numerical method. In order to measure the performance of a method, we define

$$N \equiv \frac{\text{mesh cells}}{\text{wavelength}} \text{ required for } \left| \frac{d_h}{d} - 1 \right| = 0.01 \quad (13)$$

An AP method for (11) satisfies  $N = O(\varepsilon^0)$  if  $\varepsilon \ll 1$ . After all, any discretization of (12) yields an  $N$  that is independent of  $\varepsilon$ .

### 2.1. A high-resolution Godunov method (HR)

In this section, we give an example of a non-AP method whose asymptotic behaviour was first analysed in Reference [4]. Consider a semi-discrete, high-resolution Godunov method that uses a central-difference slope reconstruction [9]. A slope limiter may also be applied, but is not needed for the purpose of this study. We use the ‘frozen’ Riemann problem (RP) for the flux solver, by which we mean that we do not account for effects of the source term in (1) when computing the interface flux.

The HR method for system (11) has modified equations given by

$$\partial_t u + \partial_x v = \frac{1}{12} h^2 \partial_x^3 v - \frac{1}{8} \frac{h^3}{\varepsilon} \partial_x^4 u + O(h^4) \quad (14a)$$

$$\partial_t v + \frac{1}{\varepsilon^2} \partial_x u + \frac{1}{\varepsilon^2} v = \frac{1}{12} \frac{h^2}{\varepsilon^2} \partial_x^3 u - \frac{1}{8} \frac{h^3}{\varepsilon} \partial_x^4 v + O(h^4) \quad (14b)$$

Consider the unresolved regime,  $h = O(1)$ . If we substitute (7) into (14a) and compare equal powers in  $\varepsilon$ , we obtain

$$\frac{1}{8} h^3 \partial_x^4 u^{(0)} = O(h^4) \quad (15a)$$

$$\partial_t u^{(0)} + \partial_x v^{(0)} = \frac{1}{12} h^2 \partial_x^3 v^{(0)} - \frac{1}{8} h^3 \partial_x^4 u^{(1)} + O(h^4) \quad (15b)$$

while (14b) yields

$$v^{(0)} = -\partial_x u^{(0)} + \frac{1}{12} h^2 \partial_x^3 u^{(0)} + O(h^4) \quad (15c)$$

We see that HR is non-AP, because (15a) is inconsistent with (12).

By substituting (15c) into (15b), it is apparent that the asymptotic behaviour is equivalent to that of

$$\partial_t u - \partial_x^2 u = - \left( \frac{1}{6} h^2 + \frac{1}{8} \frac{h^3}{\varepsilon} \right) \partial_x^4 u + O(\varepsilon, h^4) \quad (16)$$

which is this method's asymptotic modified equation. This same equation also holds in the intermediate and resolved regimes.

If the mesh is fine enough such that  $h^3 \ll \varepsilon$ , then second-order accuracy is recovered. To get an idea of the mesh requirements, the damping rate for (16) is given by

$$d_h = 4\pi^2 k^2 + 2\pi^4 k^4 h^2 \left( \frac{h}{\varepsilon} + \frac{4}{3} \right) \quad (17)$$

A good estimate for  $N$  may be obtained by ignoring the  $h^2$  term (such as in Reference [4]). Using the fact that  $h = 1/(kN)$ , we obtain

$$N = \left( \frac{50\pi^2}{\varepsilon k} \right)^{1/3} \quad (18)$$

Choosing  $k=2$  and  $\varepsilon = 10^{-5}$  requires  $N=292$  cells/wavelength to resolve the damping rate to within 1%.

Increasing the spatial order of accuracy may lower the exponent in (18), but we suspect that the resulting method will be non-AP. Note that we have shown previously that for steady linear transport, the HR method with any slope reconstruction that is independent of the source term is non-AP [9]. Another option is to replace the frozen RP by the generalized RP, which accounts for the source term when computing the flux [1]. However, the generalized RP reduces to the frozen RP as  $\Delta \tilde{t}/\tau \rightarrow 0$ , and therefore the analysis above holds in this limit. There are other fixes proposed in References [4, 7] which should also be considered, but are beyond the scope of this study.

## 2.2. Liotta, Romano, and Russo method (LRR)

The LRR method is a central scheme (extended Nessyahu and Tadmor) that is derived in Reference [2]. This method uses a uniformly non-oscillatory (UNO) procedure to compute certain derivatives; the analysis here holds for the UNO method and also any other second-order approximation. The asymptotic modified equation for the LRR method is given by

$$\partial_t u - \partial_x^2 u = \left( \frac{5}{24} h^2 - \frac{3}{128} \frac{h^3}{v\varepsilon} \right) \partial_x^4 u + O(\varepsilon, h^4) \quad (19)$$

where  $v = c_f \Delta \tilde{t}/\Delta \tilde{x}$  and  $v < 1/2$  for stability. Just as with the HR method, the  $O(h^3)$  term results in a non-AP method. Note that Reference [2, end of Section 5.1] drops  $O(h^3)$  terms

in their analysis. From Equation (19), a good estimate for  $N$  is

$$N = \left( \frac{75\pi^2}{8\varepsilon v k} \right)^{1/3} \quad (20)$$

For  $k = 2$ ,  $\varepsilon = 10^{-5}$ , and  $v = 1/2$  we obtain  $N = 210$  cells/wavelength, which is an improvement over the HR method. However, because of  $N$ 's dependence on  $v$ , in the unresolved regime the LRR method is less accurate than the HR method when  $v \lesssim 3/16$ .

### 2.3. Discontinuous Galerkin (DG)

In this section, we analyse a semi-discrete DG method. Within each cell  $j$ , the solution is approximated using a linear basis:

$$\mathbf{u}(x) = (1 - \xi)\mathbf{u}_1 + \xi\mathbf{u}_2, \quad \xi = (x - x_{j-1/2})/h \quad (21)$$

where  $\mathbf{u} = (u, v)^T$  and  $(\mathbf{u}_1, \mathbf{u}_2)$  are computed in each cell. For a linear system, DG in cell  $j$  may then be written as

$$\partial_t \mathbf{u}_1 + \frac{1}{h} [-4\mathbf{f}_{j-1/2} - 2\mathbf{f}_{j+1/2} + 3\mathbf{f}(\mathbf{u}_1) + 3\mathbf{f}(\mathbf{u}_2)] = \mathbf{s}(\mathbf{u}_1) \quad (22)$$

$$\partial_t \mathbf{u}_2 + \frac{1}{h} [4\mathbf{f}_{j+1/2} + 2\mathbf{f}_{j-1/2} - 3\mathbf{f}(\mathbf{u}_1) - 3\mathbf{f}(\mathbf{u}_2)] = \mathbf{s}(\mathbf{u}_2) \quad (23)$$

where  $\mathbf{s}(\mathbf{u})$  is the source term,  $\mathbf{f}(\mathbf{u}) = (v, u)^T$ , and the interface flux  $\mathbf{f}_{j+1/2}$  is computed via the frozen Riemann problem. More information on this particular DG implementation may be found in References [10, 11].

The asymptotic modified equation for DG is given by

$$\partial_t u - \partial_x^2 u = -\frac{1}{12} h^2 \partial_x^4 u + O(\varepsilon, h^4) \quad (24)$$

which yields  $N = 10\pi/\sqrt{3} \approx 19$  cells/wavelength, *independent* of  $\varepsilon$ . In all regimes, this equation's leading-order asymptotic behaviour is consistent with (12), and therefore, at least to  $O(h^4)$  and  $c_e = 0$ , semi-discrete DG is AP. A disadvantage of DG is that it requires twice as many unknowns per cell as the other methods in this study.

### 2.4. Numerical results for $c_e = 0$

In this section, we demonstrate that the truncation error estimates above are in good agreement with numerical results. For the semi-discrete methods, we use a predictor–corrector time integrator. The predictor may be written as

$$\frac{\mathbf{u}^{n+1/2} - \mathbf{u}^n}{\Delta t/2} = D(\mathbf{u}^n) + S(\mathbf{u}^{n+1/2}) \quad (25)$$

where the operator  $D$  corresponds to differential terms and  $S$  corresponds to the source term. For the corrector, we used a lumped-linear DG method for the source term, which requires

Table I. Results for  $c_e = 0$ ,  $\varepsilon = 10^{-5}$ ,  $u(x, 0) = \cos(2\pi x)$ ,  $v(x, 0) = 0$ , periodic domain, final time is  $t = 0.01$ .  $L_2(u)$  is the measured error, from which the order of accuracy is computed on successive meshes.  $L_2^{\text{T.E.}}(u)$  is an analytical estimate from the truncation error. Note that the fully damped solution  $u = 0$  corresponds to  $L_2(u) = 0.4765$ .

Method	Cells/wavelength	$L_2^{\text{T.E.}}(u)$	$L_2(u)$	Order
HR( $v = 0.8$ )	10	4.765e-01	4.765e-01	—
	20	4.765e-01	4.765e-01	0.00
	40	4.538e-01	4.535e-01	0.07
	80	1.509e-01	1.506e-01	1.59
LRR( $v = 0.4$ )	10	4.765e-01	4.765e-01	—
	20	4.765e-01	4.765e-01	0.00
	40	3.619e-01	3.615e-01	0.40
	80	7.763e-02	7.760e-02	2.22
DG( $v = 0.3$ )	10	6.342e-03	6.821e-03	—
	20	1.557e-03	1.587e-03	2.10
	40	3.874e-04	3.887e-04	2.03
	80	9.673e-05	9.653e-05	2.01

solving the following coupled system:

$$\frac{(\mathbf{u}^{n+1} + \mathbf{u}^*)/2 - \mathbf{u}^n}{\Delta t/2} = D(\mathbf{u}^{n+1/2}) + S(\mathbf{u}^*) \quad (26a)$$

$$\frac{\mathbf{u}^{n+1} - (\mathbf{u}^{n+1} + \mathbf{u}^*)/2}{\Delta t/2} = D(\mathbf{u}^{n+1/2}) + S(\mathbf{u}^{n+1}) \quad (26b)$$

where  $\mathbf{u}^*$  is an intermediate state. This integrator is point-implicit, L-stable, has positive amplification for all  $\Delta t$ , and is second-order accurate when  $\Delta \tilde{t}/\tau$  is small.

Table I shows results from the three methods analysed above. For each method, we Fourier transformed its asymptotic modified equation in order to analytically estimate the error with respect to the exact solution of the heat equation. This estimate is denoted as  $L_2^{\text{T.E.}}(u)$ . Also tabulated is  $L_2(u)$ , which is the measured error in  $u$  from the numerical simulation with respect to the exact solution of (1). The values of  $L_2^{\text{T.E.}}(u)$  and  $L_2(u)$  are in good agreement for all of the methods, which is a good indicator that our analysis and code implementation are correct. It also shows that the time integrator did not significantly affect the analysis results for the semi-discrete methods. The order of accuracy is computed from  $L_2(u)$ . DG shows second-order accuracy, while the other methods do not show second-order convergence until the exact solution is over-resolved.

### 3. ACCURACY FOR $c_e \neq 0$

In this section, we show the sense in which DG retains the AP property for  $c_e \neq 0$ . We use the scaling

$$\tilde{t}_{\text{ref}} = L/c_e, \quad \tilde{v}_{\text{ref}} = c_e \tilde{u}_{\text{ref}}, \quad \tilde{u}_{\text{ref}} = O(1) \quad (27)$$

so that system (1) becomes

$$\partial_t u + \partial_x v = 0 \quad (28a)$$

$$\partial_t v + \frac{1}{r^2} \partial_x u = (u - v)/(r\varepsilon) \quad (28b)$$

Past work has typically assumed that  $r = O(1)$  in the limit of small  $\varepsilon$ , so that instead of (8), results in

$$\partial_t u^{(0)} + \partial_x u^{(0)} = 0 \quad (29)$$

Another perspective is that independent of the choice of scaling,

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ r \text{ fixed}}} P_e(r, \varepsilon) = \infty \quad (30)$$

Therefore, if  $\varepsilon$  is very small (where numerical methods have difficulty), then holding  $r$  fixed tests *only* the ability to compute the advection-dominated case. If one is interested in computing solutions in only the inviscid regime ( $\varepsilon \equiv 0$ ), then fixing  $r$  as  $\varepsilon \rightarrow 0$  is valid. However, if diffusive effects are of interest, or if a method is to be accurate for all  $\varepsilon$  and  $r$ , then a more appropriate test is to hold  $P_e$  fixed as  $\varepsilon \rightarrow 0$ . One may then test the diffusion-dominated case by choosing  $P_e$  small. No generality is lost, since one may select a large  $P_e$  in order to test the advection-dominated case.

A fixed  $P_e$  implies that  $r = O(\varepsilon)$ , which changes the asymptotic analysis such that the leading order solution is given by (8). This is another example of a diffusive scaling [7]. Another consequence of the scaling is that one may approximate  $P_e$  in (4) as  $r/\varepsilon$ , but this approximation is not required. In fact, we prefer the form in (4), because it guarantees that  $|r| < 1$  (bounded solutions) for any  $P_e > 0$  and  $\varepsilon > 0$ .

The asymptotic modified equation for the HR method is given by

$$\partial_t u + \partial_x u - \frac{1}{P_e} \partial_x^2 u = -\frac{h^2}{12} \left( \frac{2}{P_e} \partial_x^4 - \partial_x^3 \right) u - \boxed{\frac{1}{8} \frac{h^3}{r} \partial_x^4 u} + O(\varepsilon, h^4) \quad (31)$$

while for DG, we obtain

$$\partial_t u + \partial_x u - \frac{1}{P_e} \partial_x^2 u = -\frac{1}{36P_e} (3h^2 \partial_x^4 + 2rh^3 \partial_x^5) u - \boxed{\frac{1}{72} rh^3 \partial_x^4 u} + O(\varepsilon, h^4) \quad (32)$$

Note that if we account for the scaling differences between (27) and (10), and set  $c_e = 0$ , then (31) reduces to (16) and (32)–(24). The relations above hold for all mesh resolution regimes and for  $r = O(1)$  or  $O(\varepsilon)$ .

The important difference between HR and DG is contained in the boxed terms of (31) and (32). When  $r = O(1)$ , the boxed terms are not a problem for either method and both are second-order accurate. Keep in mind that

$$\lim_{\substack{P_e \rightarrow \infty \\ \varepsilon \text{ fixed}}} r(P_e, \varepsilon) = \pm 1$$



so that at a given  $\varepsilon$  and large enough  $P_e$ , second-order accuracy will be observed for either method.

But when  $r = O(\varepsilon)$ , we have that  $r = \varepsilon P_e + O(\varepsilon^3)$ , so the boxed term in (31) may be approximated as

$$\frac{1}{8P_e} \frac{h^3}{\varepsilon} \partial_x^4 u$$

As in the  $c_e = 0$  case, it is then obvious that the term above will dominate the diffusive effects when  $h^3 \gg \varepsilon$ . The above relation is the leading order solution when  $h = O(1)$ , so that HR is non-AP. A similar result may be derived for the LRR method.

In contrast to HR, the boxed term in (32) behaves as  $h^3 \varepsilon$ . Therefore, DG maintains second-order accuracy for all mesh regimes and is AP. Yet another argument is to compare (32) with a second-order discretization of (2). A semi-discrete discretization of (2), using a central-difference slope reconstruction, the upwind flux solver, and a three-point central discretization for the diffusion term has a modified equation given by

$$\partial_t u + \partial_x u - \frac{1}{P_e} \partial_x^2 u = \frac{1}{12} h^2 \partial_x^3 u - \frac{1}{12P_e} h^2 \partial_x^4 u - \boxed{\frac{1}{8} h^3 \partial_x^4 u} + O(h^4) \quad (33)$$

The boxed term here is similar to that as in Equation (32), but without the factor of  $r$ . But because  $|r| < 1$ , DG has similar accuracy.

The analysis above strongly suggests that numerical tests should be run at a fixed  $P_e$ . For small values of  $P_e$ , only DG should be second-order accurate, but for a given  $\varepsilon$  and large enough  $P_e$ , all of the methods in this study should appear second-order. Figure 1 compares  $L_2$ -errors from the DG and LRR methods for three values of the Peclet number. Each plot shows results that are roughly in the unresolved ( $\varepsilon = 10^{-5}, 10^{-4}, 10^{-3}$ ), intermediate ( $\varepsilon = 0.02$ ), and resolved ( $\varepsilon = 10^5$ ) regimes. The problem's initial condition was  $u(x, 0) = \cos(2\pi x)$ ,  $v(x, 0) = u(x, 0)$ , with periodic boundary conditions. The final time was chosen so that the equilibrium wave propagates 1 wavelength. The DG method shows second-order accuracy, independent of  $\varepsilon$ . Both methods perform similarly in the intermediate and resolved regimes, but the LRR method generally does poorly in the unresolved regime. As predicted, if  $P_e$  is large enough, for a given  $\varepsilon$  (roughly when  $P_e \varepsilon = O(1)$ ), second-order accuracy is recovered by LRR in the unresolved regime. The results of the HR method (not shown) are very similar to the those of LRR.

#### 4. CONCLUSIONS

We have shown that semi-discrete DG is asymptotic preserving (AP) for a model problem. To discriminate between AP and non-AP methods, we have argued that one should fix the Peclet number in the limit of vanishing relaxation time. To extend this concept to more general systems, in the limit of the relevant small parameter, one should fix whatever dimensionless parameters characterize the near-equilibrium limit. In other work [10, 11], we have obtained good results for DG for non-linear systems, such as for the Broadwell model of gas kinetics and problems in radiation hydrodynamics. The failure of non-AP methods is often the result

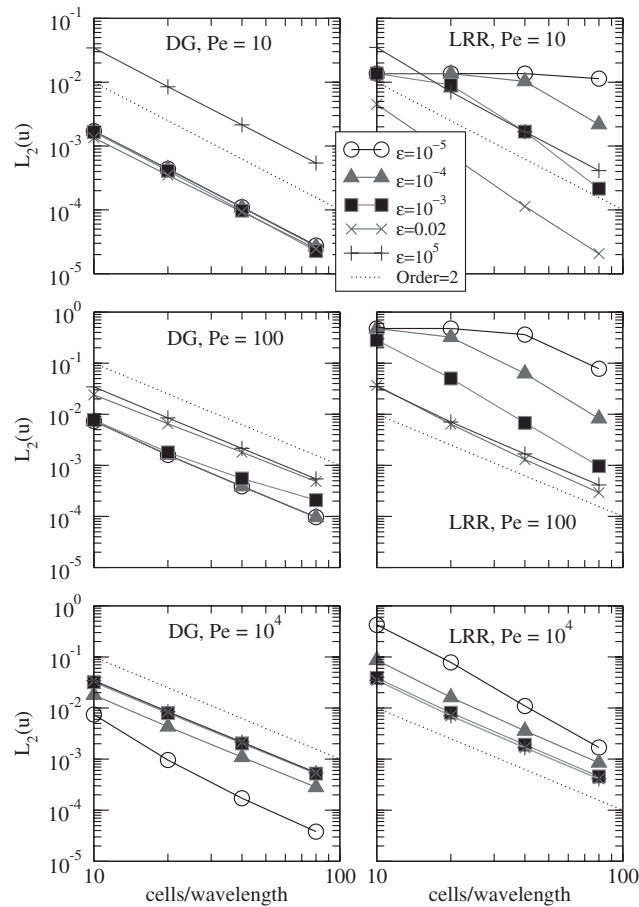


Figure 1. DG( $\nu = 0.3$ ) and LRR( $\nu = 0.4$ ) errors for various Peclet numbers.

of higher-order terms in  $h$ . Moreover, the terms that cause failure may be traced back to the discretization and possibly remedied (e.g., see References [4, 7]). This analysis is left for future work.

#### ACKNOWLEDGEMENTS

The authors would like to thank Jeffrey A. Hittinger for discussions and the anonymous referees for their many insightful comments. This work was performed under the auspices of the U.S. Department of Energy by Los Alamos National Laboratory under Contract W-7405-ENG-36. LAUR 01-1301 (revised).

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